

Physics from scratch.

Letter on M. Tegmark's "The Mathematical Universe"

Antonio N. Bernal¹, Miguel Sánchez¹, Francisco José Soler Gil²

¹ *Departamento de Geometría y Topología*

Universidad de Granada, Spain

² *Institut für Philosophie*

Universität Bremen, Germany

sanchezm@ugr.es, soler@uni-bremen.de

Abstract. In a recent article, M. Tegmark [6] poses the hypothesis that our known universe is a “baggage free” mathematical structure among many other possible ones, which also correspond to other physical universes —Mathematical Universe Hypothesis, MUH. Naturally, questions arise, such as how to obtain the physical properties of our world from the mathematical structure, or how many possibilities exist for a Universe minimally similar to ours.

In this letter we present some results which can be regarded as a strengthening of MUH, as they give some hints on the derivation of spacetime in current physics from a baggage free mathematical structure.

Concretely, we argue that the set of mathematical structures which can be interpreted as a description of a spacetime is drastically reduced, if one admits some natural postulates on minimal symmetry. Furthermore, the apparently very particular form of classical Galilei-Newton and relativistic spacetimes, is not arbitrary and cannot be regarded as “two possibilities among arbitrarily many others”. In fact, such theories are determined by a single mathematical structure which only permits four possible types of spacetimes. Finally, we show how the minimal postulates on symmetry can be endowed with a simple physical interpretation, i.e., they acquire “baggage” in a natural way.

1 Introduction

In his recent article [6] (see also [7]), Tegmark explored the implications of the *Mathematical Universe Hypothesis* (MUH) which, essentially, claims the full identification of both notions, “mathematical existence” and “physical existence”. According to MUH, our universe is really a mathematical structure, and conversely: each consistent mathematical structure would be also a physically real universe. With this proposal, Tegmark argues the possibility to answer some of the deepest questions about the foundations of physics, such as Wigner’s *why are so effective mathematics in natural sciences?* or Wheeler’s *why these equations and no others?* Tegmark’s picture of the world is then a physical reality with no free parameters and no arbitrary elements of any kind: all non-contradictory possibilities are real. Consequently, nothing is contingent, and no open question remain.

From the very beginning, the authors of the present article must recognize that they are not very enthusiastic about these ideas. For the simple reason that Tegmark’s proposal leaves important elements unexplained such as, for example, the elusive reality of the time (how to do justice to the details of our experience of the “now” and the temporal becoming of reality in the context of a tenseless mathematical world?). Certainly, such troubles are shared by our major physical theories—General Relativity, in the case of time. But most people makes no claims in order to identify these theories with the totality of physical reality.

Nevertheless, Tegmark’s proposal is very attractive, and worth exploring. Therefore, in the next pages, we develop some ideas which could support MUH. They rely on a postulational basis for spacetime introduced from a different viewpoint by two of the authors (ANB & MS) and M. López [1]. Concretely:

1. We postulate that the elements of a certain mathematical structure, with no physical baggage, satisfy certain symmetry (Section 2). As a consequence, four and only four variants of this structure can exist (Section 3).
2. Two of these variants correspond to a generalization of the Galilei-Newton theory and General Relativity (Section 4). This stresses:
 - The concrete form of the physical laws of our universe can be determined by symmetry conditions on a purely mathematical underlying structure.
 - To postulate a minimum of symmetry in such an underlying mathematical structure, drastically limits the number of possible physical scenarios.
3. The conditions of symmetry and other aspects can be easily endowed with physical meaning (Section 5). This suggests that many properties perceived by physical observers may emerge from the mathematical properties of the underlying structure.

For simplicity, we focus on the structure of classic spacetime. In the case that our universe were a mathematical structure, such a structure would also be responsible for the quantum aspects of the world. Nevertheless, to consider the classic case is enough, in order to have a taste of the dependence of many particularities of the physical world on purely mathematical features. This is precisely the point which makes MUH so suggestive and attractive.

2 A nice mathematical structure

Consider the following simple mathematical background structure (MB):

MB. The pair (M, S) , where M is a smooth connected 4-manifold, with differentiable structure denoted as \mathcal{D} , and $S : M \rightarrow \mathcal{P}(\mathcal{D}), p \mapsto S_p$ is a map (with codomain the set of parts of¹ \mathcal{D}) which satisfies:

S_p is non-empty, and all the charts in S_p are defined on some neighborhood of p .

Notice that each element of \mathcal{D} is a coordinate chart $(U, \phi), \phi : U \subseteq M \rightarrow \mathbb{R}^4$. The (ordered) coordinates will be denoted as (x^0, x^1, x^2, x^3) . Now, let us enrich MB with a symmetry postulate (SP):

SP. Any two charts $O \equiv (x^0, x^1, x^2, x^3), \tilde{O} \equiv (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ in S_p satisfy:

$$\partial_{x^0} \tilde{x}^0|_p = \partial_{\tilde{x}^0} x^0|_p, \quad \partial_{x^j} \tilde{x}^i|_p = \partial_{\tilde{x}^i} x^j|_p, \quad \forall i, j \in \{1, 2, 3\}. \quad (2.1)$$

Notice that this hypothesis assumes two independent symmetries between the different charts selected for each p : a symmetry for the 0-th coordinate and a symmetry for the set of the other three coordinates.

Finally, we will restrict our attention to the *smooth class* **SC** of mathematical structures where the symmetry assumption SP fits smoothly with the ambient manifold M . The precise formulation of this class becomes trivial when the consequences of SP are analyzed, as we will see below.

3 The set of resulting mathematical structures

Previous elements MB, SP and SC correspond to postulates in an approach for the study of spacetime in [1]. This approach develops a different viewpoint (in Tegmark's terms, from the frog to the bird viewpoint²), but its careful mathematical analysis is applicable now.

Let S^1 be the circumference obtained from the extended real line $[-\infty, \infty]$ by collapsing $\pm\infty$ to a single point ω . As proven in full detail in [1]:

Under the background MB, the symmetry postulate SP allows to attach to the tangent space $T_p M$ at p both, some $k(p) \in S^1$ and one among four concrete linear structures.

More precisely, these four structures are:

1. If $k(p) \in (-\infty, 0)$: a Lorentzian scalar product on $T_p M$. (The value of $k(p)$ corresponds to the normalization of $\partial_{x^0}|_p$ for all the charts in S_p .)
2. If $k(p) = \omega$: a (non-zero) linear form $\Omega_p \in T_p M^*$ ($\Omega_p : T_p M \rightarrow \mathbb{R}$) and an Euclidean scalar product h_p on the kernel of Ω_p .
3. If $k(p) = 0$: a (non-zero) tangent vector $Z_p \in T_p M$ and an Euclidean scalar product h_p^* on the kernel of Z_p in $T_p M^*$.

¹i.e., \mathcal{D} is the set of all the smooth coordinate charts of M and $\mathcal{P}(\mathcal{D})$ is the set of all the subsets of \mathcal{D} .

²See also [4] for additional developments which may be useful in this section.

4. If $k(p) \in (0, \infty)$: an Euclidean scalar product on $T_p M$.

As explained in [1], the restriction to the smooth class SC consists simply in the restriction to the case where the function $k(p)$ (and the corresponding linear structure), vary smoothly with p .

So far, we have considered only a purely mathematical structure or, in Tegmark’s terminology, a bird viewpoint of (a piece of) reality. Next, we will descend from this Platonic world, looking for the physical meaning of the mathematical structure.

4 The (very small) family of classical spacetimes

The a priori deduction of the physical properties of a mathematical structure, is a serious challenge for defenders of MUH. Tegmark poses the problem as follows [6, Section III]:

Suppose we were given mathematical equations that completely describe the physical world, including us, but with no hints about to interpret them. What would we do with them? Specifically, what mathematical analysis of them would reveal their phenomenology, i.e., the properties of the world that they describe as perceived by observers? [...] This is in my opinion one of the most important questions facing theoretical physics, because if we cannot answer it, then we cannot test a candidate TOE by confronting it with observation. [...] By construction, the only tools at our disposal are pure mathematical ones, so the only way in which familiar notions and interpretations (“baggage”) can emerge are as implicit properties of the structure itself that reveal themselves during the mathematical investigation.

Clearly, the most rigorous way to descend from the bird’s viewpoint to that of the frog (the viewpoint of an observer who lives in the world and is part of it), is to start from the purely mathematical analysis and then to detect the emergency of the physical concepts —as implicit properties of the analyzed structures. Nevertheless, this ideal procedure is not only difficult but also vague, as the meaning of “a physical property implicit in a mathematical structure” is by no means clear.

Thus, instead of following this path, our starting point in this section will be the already known physics. The comparison between the previously introduced mathematical structures and the known theories of spacetime is trivial. However, as we will see, not all the results of the comparison are trivial. Some of them can be interpreted as a strengthening of the intuitions underlying MUH —a step towards understanding the emergency of physical properties from certain aspects of mathematical structures.

Let us start with the most obvious one:

1. Structures with $k(p) \in (-\infty, 0)$, for all $p \in M$. Recall that in this case a scalar Lorentzian product at each tangent space is obtained. That is, one has a Lorentzian metric on a 4-manifold —the ambient for spacetime of classical General Relativity. Additionally, all the orthonormal basis at each $T_p M$ can be chosen normalized so that the metric on the first vector is equal to $k(p)$. Putting $c(p)^2 = -k(p)$ a closer analogy to General Relativity is obtained.

2. $k \equiv \omega$. One has a non-vanishing 1-form Ω on M and a Riemannian metric h on the fiber bundle obtained as the kernel of Ω . This is called a *Leibnizian structure* in [1], and it is studied exhaustively in [2]. It is a clear generalization of Galilei-Newton spacetime, where Ω is an exact form (the differential $\Omega = dT$ of the “absolute time” T), and a flat connection ∇ which parallelizes h is assumed to exist.
3. $k \equiv 0$. This case is mathematically dual and analogous to the previous one, even though it does not have classical analog.
4. $k(p) \in (0, \infty)$, for all $p \in M$. Reasoning as in the first case, now one has a (positive-definite) Riemannian metric on the manifold M . Notice that, according to SP, one only assumes the existence of independent symmetries between, on one hand, the first coordinates, and, on the other, the sets composed of the other three coordinates (formula (2.1)). This does not, however, exclude the existence of additional symmetries between all the coordinates —which is the present case. There are slight strengthenings of SP which would forbid this case (say, assuming that the members of the first equality in (2.1) are positive). Nevertheless, according to MUH, our more general version of SP must be also taken into account —and suggests interesting possibilities.

Now, let us emphasize a first interesting conclusion: the relation between the Galilei-Newton and relativistic structures of spacetime. They are singled out as members of a family of only four elements —the mathematical structures compatible with MA, SP, SC. This suggests that classical and relativistic physics are not two arbitrary physical scenarios within a set of innumerable alternatives.

Considering the most radical possibility, the introduced mathematical structure might be the bases from which classical spacetimes arise. Of course, there are more general possibilities: consider, for example, the case of n coordinates and independent symmetries for the first $m < n$ ones. Nevertheless, these scenarios may be so different that they might not contain observers —say, self-aware structures in a sense minimally similar to us. So, Wheeler’s question in this context (why these spacetimes and no others?) would admit a first answer: there were not so many possible alternatives. To which extent this conclusion is sound must be examined carefully but, in any case, it is clearly worth taking into account.

Assuming the conclusion as a working hypothesis, the following appealing item is the key role played by the symmetry condition SP. Essentially, this symmetry seems to be so powerful that it generates all the spacetimes. This seems to agree with Tegmark’s claim:

[...] when studying a mathematical structure S to derive its physical phenomenology (the “inside view”), a useful first step is finding its symmetries [...]

Thus, next we will study next the physical meaning of our spacetime-generating condition of symmetry —beyond a purely mathematical element.

5 The physical meaning of the symmetry condition

Let us reformulate PS as follows:

For each point $p \in M$ there exists a non-empty subset S_p of coordinate charts or *spacetime observations around p* , the first coordinate $t \equiv x^0$ of each chart labelled as the *time coordinate* and the other three $x^i, i = 1, 2, 3$ as the *spatial coordinates*. Then, given two such charts $O \equiv (t, x^1, x^2, x^3), \tilde{O} \equiv (\tilde{t}, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ in S_p there exist a symmetry at p between the timelike coordinates:

$$\partial_t \tilde{t}|_p = \partial_{\tilde{t}} t|_p \quad (5.1)$$

and also between the spatial coordinates:

$$\partial_{x^j} \tilde{x}^i|_p = \partial_{\tilde{x}^i} x^j|_p, \quad \forall i, j \in \{1, 2, 3\}. \quad (5.2)$$

This condition was introduced above as a mere element of the mathematical structure. Nevertheless, as the names introduced now suggest (and proved in full detail in [1]), such an element can be endowed with a simple physical meaning. Summing up:

- The set S_p of privileged charts is just the set of observations carried out by the *standard observers* of the theory (say, at least infinitesimally standard observers at p : inertial, freely falling, using ideal instruments, selected by some sort of ether...)
- The symmetry (5.1) between the timelike coordinates ensures that the measures of time cannot privilege one of the two standard observers over the other one. That is, the \tilde{O} -time, measured with the O clock, goes by as the O -time, measured with the \tilde{O} clock — O and \tilde{O} measure at p using the same “fundamental unit of time”.
- Analogously, the second symmetry (5.2) ensures the inexistence of privileges for the spatial coordinates or, equally, the use of the same fundamental space units by O and \tilde{O} .

Summing up, the maximum number of classic spacetime theories seem to be just four, and: (a) this is obtained from a symmetry mathematical postulate on a 4-manifold, and (b) from this postulate, a simple baggage interpretation emerges. These two facts agree with Tegmark’s picture derived from MUH.

6 Conclusion

We repeat that the authors do not agree with the full identification of both, reality and mathematical structures, which lies in the core of MUH. Long discussions on this topic are possible, and we have already mentioned the problem of our experience of time —real or illusory, in any case unexplained³. This is only an example of our reluctance among others: the unexplained experience of subjective perceptions (“qualia”), the apparent superfluous character of quantum level 3 multiverses, the unclear falsability⁴ of the theory... However, to dispute this topic is not our purpose.

³See [3, 5] about the difficulties of explaining the human experience of time starting at the “tenseless” existence of the reality.

⁴One of the authors (FJSG) thinks that there could be in principle uncountably many universes similar to ours, obtained for example by allowing the “constants of nature” to be variable. If this were true, the generic classic universes would be those with “almost constants” of nature (...or they could be universes still more irregular, perhaps). As such almost constants are not found, we would be living in a very particular universe. And this would not fit well in MUH. A possible answer would be that even a minimal change in the constants of the world precludes the existence of observers. But, in this case, is MUH testable at all?

Anyway, we have seen along the present letter that, at least in the case of classic spacetimes, the relation between the mathematical formalism and the physical meaning fits the expectations of MUH: starting at the symmetries of a concrete mathematical structure, the physical properties perceived by the (frog) observers emerges.

This reinforces MUH. Perhaps we would not bet that the universe is as Pythagorean as Tegmark suggests, but the existence of a strong Pythagorean component is undisputable.

Acknowledgments

MS has been partially supported by Spanish MEC-FEDER Grant MTM2007-60731 and Regional J. Andalucía Grant P06-FQM-01951.

References

- [1] A.N. Bernal, M. López, M. Sánchez: Fundamental units of length and time, *Found. Phys.* **32** (2002), no. 1, 77–108.
- [2] A.N. Bernal, M. Sánchez: Leibnizian, Galilean and Newtonian structures of space-time, *J. Math. Phys.* **44** (2003), no. 3, 1129–1149.
- [3] W.L. Craig: *The tenseless theory of time: A critical examination*, Dordrecht: Kluwer, (2001).
- [4] M. Sánchez: *On the foundations and necessity of gauge theories*, preprint.
- [5] F.J. Soler Gil: Die präsentistische Auffassung der Zeit im Kontext der Relativitätstheorien und der Quantenkosmologie von James Hartle und Stephen Hawking: ein Vergleich, *Philosophia naturalis* **44/1** (2007) 114-180.
- [6] M. Tegmark: The Mathematical Universe, *Found. Phys.*, **38** (2008) 101-150.
- [7] M. Tegmark: Is “the Theory of Everything” Merely the Ultimate Ensemble Theory? *Annals of Physics*, **270** (1998) 1-51